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Corrigendum

Corrigendum to "Harmonic analysis on perturbed Cayley Trees" [J. Funct. Anal. 261 (3) (2011) 604–634]

Francesco Fidaleo

Dipartimento di Matematica, Università di Roma Tor Vergata, Via della Ricerca Scientifica 1, Roma 00133, Italy

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Abstract

Due to the boundary effects, the standard definition of the *integrated density of the states* (i.d.s. for short) used in [F. Fidaleo, Harmonic analysis on perturbed Cayley Trees, J. Funct. Anal. 261 (3) (2011) 604–634], does not work for nonamenable graphs like Cayley Trees and density zero perturbations of those. On the other hand, Proposition 2.3 in the previous mentioned paper works under the right definition we are going to describe, and which is useful for all the applications. For the sake of completeness and the convenience of the reader, we also show that both the definitions coincide in the amenable case.

Keywords: Harmonic analysis on Cayley Trees; Bose Einstein condensation; Integrated density of the states

1. Integrated density of the states

In the present note we use the notations of [1]. In addition, we use the following definition of amenability. Let $K \subset G$ be a subgraph of a graph G which is always supposed to be connected. Define for $r \in \mathbb{N}$,

$$\partial_r K := \{x \in VG \backslash VK \mid \operatorname{dist}(x, K) \leqslant r\}.$$

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E-mail address: fidaleo@mat.uniroma2.it.

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Fix an exhaustion $\{K_n\}_{n\in\mathbb{N}}$ of G. Such an exhaustion is said to be *amenable* if

$$\lim_n \frac{|\partial_r K_n|}{|VK_n|} = 0, \quad r = 1, 2, \ldots. \tag{1.1}$$

The graph G is said to be *amenable* if it admits an amenable exhaustion. It is well known that the Cayley Tree $\mathbb{G}^q$ is nonamenable whenever $q > 2$.

Let G be any (connected) graph equipped with an exhaustion $\{K_n\}_{n\in\mathbb{N}}$ which is kept fixed in the sequel. Consider on $\mathcal{B}(\ell^2(VG))$ the state

$$\tau_n := \frac{1}{|VK_n|} \operatorname{Tr}_n(P_n \cdot P_n),$$

P_n being the selfadjoint projection onto $\ell^2(VK_n)$. Define for a bounded operator B,

$$\tau(B) := \lim_n \tau_n(B), \quad B \in \mathcal{D}_\tau, \tag{1.2}$$

where the domain $\mathcal{D}_\tau$ is precisely the linear subspace of $\mathcal{B}(\ell^2(VG))$ for which the limit in (1.2) exists. In addition, define for a bounded selfadjoint operator B,

$$\tau^B\big(f(B)\big) := \lim_n \tau_n\big(f(P_n B P_n)\big), \quad f \in C\big(\sigma(B)\big), \tag{1.3}$$

provided such a limit exists. The domain $\mathcal{D}_{\tau^B} \subset C^*(B)$ is precisely the linear subspace of the unital C^*-algebra $C^*(B) \subset \mathcal{B}(\ell^2(VG))$ generated by B, for which the limit in (1.3) exists. Notice that the definition of τ, τ^B depends on the chosen exhaustion which we keep fixed during the analysis.

Suppose now that $\mathcal{D}_{\tau^B} = C^*(B)$. Then it provides a state on $C^*(B)$ and, by the Riesz–Markov Theorem, a Borel probability measure μ_B on $\sigma(B)$. Thus, there exists a right continuous, increasing, positive function F_B satisfying

$$F_B(x) = 0, \quad x < \min\sigma(B); \qquad F_B(x) = 1, \quad x \geqslant \max\sigma(B),$$

such that

$$\mu_B\big((-\infty, x]\big) = F_B(x), \quad x \in \mathbb{R}.$$

Definition 1.1. The previous described cumulative function F_B is called *the integrated density of the states* associated to B, provided it exists for the chosen exhaustion.

Let A be the adjacency operator of the Cayley Tree $\mathbb{G}^q$. Proposition 2.3 of [1] assures the existence of the i.d.s. for such an operator. It becomes

Proposition 1.2. $\mathcal{D}_{\tau^A} = C^*(A)$ *and*

$$\tau^A\big(f(A)\big) = \int f\big(\|A\| - x\big)\,\mathrm{d}F(x),$$

where the Laplace transform of F *is the function given in* (2.5) *of* [1].

Proof. The same as Proposition 2.3 in [1], by taking into account that $A_{X_n} = P_n A_X P_n$. □

According to the present notations, Proposition 2.5 of [1] assumes the form

Proposition 1.3. *Let Y be a negligible perturbation of the tree X. Then $\mathcal{D}(\tau^{A_Y}) = C^*(A_Y)$, and*

$$\tau^{A_Y}\big(f(A_Y)\big) = \tau^{A_X}\big(f(A_X)\big). \tag{1.4}$$

Proof. Although the estimation (2.10) is correct and leads to the result also in the present situation, the proof considerably simplifies as follows. The Krein Formula

$$(P_n A_Y P_n)^k - (P_n A_X P_n)^k = \sum_{l=1}^{k} (P_n A_Y P_n)^{k-l} D_{XY} (P_n A_X P_n)^{l-1},$$

together with the analogous of (2.10) of [1]

$$\begin{aligned}\big|\tau_n\big((P_n A_Y P_n)^r D_{XY} (P_n A_X P_n)^s\big)\big| &\leqslant \|A_Y\|^r \tau_n\big((P_n A_X P_n)^s D_{XY}^2 (P_n A_X P_n)^s\big)^{1/2} \\ &\leqslant \|A_Y\|^r \|A_X\|^s \tau_n\big(D_{XY}^2\big)^{1/2},\end{aligned}$$

leads to the assertion, by taking into account that Y is a density zero perturbation of X. □

2. Amenable case

Due to the boundary effects, it is in general unclear whether the existence of the i.d.s. F_B for B implies the existence of that $F_{f(B)}$ of $f(B)$, where f is a continuous real function on the real line. In addition, it is unclear whether we can express $F_{f(B)}$ in terms of F_B, provided both exist. We now prove that this is not the case when the graph is amenable and the operator B has finite propagation.

Let G be an amenable graph equipped with an amenable exhaustion $\{K_n\}_{n\in\mathbb{N}}$. Fix a finite propagation bounded selfadjoint operator $B \in \mathcal{B}(\ell^2(VG))$. In some pivotal situations (see [2] and the literature cited therein) it can be proven that $\mathcal{D}_{\tau^B} = C^*(B)$. We assume that this is the case.

Theorem 2.1. *Let G be an amenable graph and $B \in \mathcal{B}(\ell^2(VG))$ a finite propagation selfadjoint operator with i.d.s. F_B. For every bounded continuous real functions on the real line f, g we get*

$$\tau^{f(B)}\big(g\big(f(B)\big)\big) = \tau^B\big((g\circ f)(B)\big) = \tau\big((g\circ f)(B)\big). \tag{2.1}$$

Thus

$$F_{f(B)}(y) = \int \chi_{f^{-1}((-\infty,y])}(x)\,\mathrm{d}F_B(x). \tag{2.2}$$

Proof. To prove (2.1), it is enough to show $\tau^B(F(B)) = \tau(F(B))$ for each bounded continuous function F on the real line. By a standard approximation argument, we can reduce the matter to monomials $F(x) = x^k$, $k \in \mathbb{N}$. As the involved operators are finite rank (due to the presence of projections onto finite dimensional subspaces), we can freely use the standard trace Tr on $\mathcal{B}(\ell^2(VG))$. Let r be the propagation length of B. We get

$$P_n B^k = (P_n B)^k + \sum_{l=1}^{k-1} (P_n B)^l Q_{n,r} B^{k-l},$$

where $Q_{n,r}$ is the orthogonal projection onto $\ell^2(\partial_r K_n)$. We then get

$$\begin{aligned}\left|\tau_n\left(P_n B^k\right) - \tau_n\left((P_n B P_n)^k\right)\right| &\leqslant \sum_{l=1}^{k-1} \frac{1}{|VK_n|} \left|\operatorname{Tr}\left(B^{k-l}(P_n B)^l Q_{n,r}\right)\right| \\ &\leqslant \|B\|^k (k-1) \sqrt{\frac{|\partial_r K_n|}{|VK_n|}} \to 0\end{aligned}$$

by the amenability assumption (1.1).

Once having established (2.1), we get for every continuous real functions f, g on the real line,

$$\begin{aligned}\int g(y)\,\mathrm{d}F_{f(B)}(y) &= \tau^{f(B)}\left(g\left(f(B)\right)\right) = \tau\left((g \circ f)(B)\right) \\ &= \tau^B\left((g \circ f)(B)\right) = \int g\left(f(x)\right) \mathrm{d}F_B(x).\end{aligned}$$

The assertion follows as, in our situation $\int g(y)\,\mathrm{d}F_{f(B)}(y) = \int g(f(x))\,\mathrm{d}F_B(x)$ is equivalent to (2.2). □

References

[1] F. Fidaleo, Harmonic analysis on perturbed Cayley Trees, J. Funct. Anal. 261 (3) (2011) 604–634.

[2] F. Fidaleo, D. Guido, T. Isola, Bose Einstein condensation on inhomogeneous amenable graphs, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 14 (2011) 149–197.